EFFECT OF PARTICLES SUSPENDED IN A FLUID FLOW ON THE DECAY OF ISOTROPIC TURBULENCE

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Dynamic equations have been obtained for the two-point double correlations of the fluctuation velocities of a fluid and the particles suspended in it at low volume concentrations of the solid phase. In the case of uniform isotropic turbulence these equations can be considerably simplified. The final period of decay of isotropic turbulence has been studied in detail. At this stage in the case of high-inertia particles the inhomogeneous-fluid turbulence is similar to the turbulence of a homogeneous fluid (without particles) in the sense that the presence of the particles affects only the fluctuation energy but leaves unchanged the spatial scales of turbulence and the spatial energy spectrum function. The suspended particles lead to exponential damping of the turbulent pulsations.

Little theoretical information is available on the hydrodynamics of a suspension of fine particles in a turbulent liquid or gas. Research has been mainly confined to the behavior of the individual particles in a given turbulence field [1]. The problem of the turbulent motion of the mixture as a whole has been examined by Barenblatt [2], who derived the equations of motion of the mixture, using Kolmogorov's hypothesis to close them. Hinze [3] has also attempted to derive equations for turbulent pulsations of the mixture. However, as Murray showed [4], Hinze's equations contradict Newton's third law.

The effect of suspended particles on the turbulence of a two-phase flow is governed by the noncorrespondence of the local velocities of the particles and the medium. The forces of resistance to the motion of the particles relative to the fluid lead to additional dissipation of fluctuation energy and decay of turbulence [2]. On the other hand, if the averaged velocities of particles and medium do not correspond, the suspended particles may also have a destabilizing effect [5, 6], causing energy transfer from the averaged to the pulsating motion. Below we shall consider the case where the averaged velocities of the two phases coincide, i.e., we shall deal only with the first of the two above-mentioned effects.

1. Formulation of the problem. Following Barenblatt [2], we write the equation of motion of the fluid and the suspended particles in the form

$$d_1(1-\rho)\left(\frac{\partial}{\partial t}+v_j\frac{\partial}{\partial x_j}\right)v_i = -\frac{\partial p^{(1)}}{\partial x_i}+\frac{\partial \tau_{ji}^{(1)}}{\partial x_j}-d_1(1-\rho)g_i-f_i,$$

$$d_2\rho\left(\frac{\partial}{\partial t}+w_j\frac{\partial}{\partial x_j}\right)w_i = -\frac{\partial p^{(2)}}{\partial x_i}+\frac{\partial \tau_{ji}^{(2)}}{\partial x_j}-d_2\rho g_i+f_i. \qquad (1.1)$$

Here v_i and w_i are the velocities of fluid and particles, and d_1 and d_2 are the densities of the fluid and the particle material, ρ is the volume concentration of the particles, g_i are the components of gravitational acceleration, $p^{(1)}$, $p^{(2)}$ and $\tau_{ji}^{(1)}$, $\tau_{ji}^{(2)}$ are the pressures and viscous stress tensors for fluid and particles, respectively, f_i is the interaction force operating between particles and fluid normalized for unit volume of the mixture.

We shall assume that d_1 = const (incompressible fluid), d_2 = const, and concentration $\rho \ll 1$. The latter assumption enables us to set $d_1(1-\rho) \approx d_1$ in the first of equations (1.1). In view of the smallness of ρ we can neglect the effect of particle interaction, i.e., take $\tau_{ji}^{(2)}$ and $p^{(2)}$ equal to zero. Setting

$$\tau_{ij}^{(1)} = \mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right), \qquad p^{(1)} = p,$$

where μ is the viscosity of the fluid, instead of (1.1) we get

$$\left(\frac{\partial}{\partial t}+v_j\frac{\partial}{\partial x_j}\right)v_i = -\frac{1}{d_1}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 v_i}{\partial x_j\partial x_j}-g_i-\frac{f_i}{d_1},$$

$$\nu=\frac{\mu}{d_1}, \quad \left(\frac{\partial}{\partial t}+w_j\frac{\partial}{\partial x_j}\right)w_i = \frac{\varkappa}{d_1\rho}f_i-g_i, \qquad \varkappa=\frac{d_1}{d_2}. \qquad (1.2)$$

The continuity equation for the fluid and the mass balance equation for the particles, given the above assumption that $\rho \ll 1$, assume the form

$$\frac{\partial v_j}{\partial x_j}=0, \qquad \frac{\partial \rho}{\partial t}+\frac{\partial \rho w_j}{\partial x_j}=0. \qquad (1.3)$$

As usual, we consider the turbulence uniform in the sense that, firstly, the averaged characteristics of the motion in the region in question do not depend on the coordinates and time, and, secondly all the two-point correlations depend only on the vector of the distance between the points, but not on the location of these points.

We introduce the fluctuations in velocity, pressure and particle concentration

$$v_i = \langle v_i\rangle + v_i', \quad w_i = \langle w_i\rangle + w_i',$$

$$p = \langle p\rangle + p', \quad \rho = \langle \rho\rangle + \rho'.$$

The sign $\langle\ \rangle$ denotes averaging with respect to time or a small physical volume. In accordance with the usual method [7], we assume that these time or space averages are identical with the probability means.

Finally, we assume that for averaged motion there is no force $\langle f_i\rangle$, i.e., $\langle v_i\rangle = \langle w_i\rangle$. This assumption is necessary, in particular, in studying isotropic turbulence. From the physical viewpoint, it is, generally speaking, equivalent to the assumption that the forces of gravity are small compared with the viscous and inertia forces.

2. The dynamic equations for the correlations. It is easy to see that for the fluctuations v_i' the first of equations (1.3) holds; from the second of equations (1.3) it follows that

$$\langle\rho\rangle\frac{\partial w_j'}{\partial x_j} = -\frac{\partial \rho'}{\partial t}-\langle w_j\rangle\frac{\partial \rho'}{\partial x_j}-\frac{\partial \rho' w_j'}{\partial x_j}.$$

Neglecting ρ in the first of equations (1.1) and in the continuity equation for the fluid means assuming that correlations of the type

$$\langle\rho' v_i' v_j'\rangle, \quad \langle\rho' v_i' v_j' v_k'\rangle, \quad \langle\rho' w_i' v_j'\rangle, \text{ etc.}$$

are small compared with the correlations of the same order for the velocities v_i' and w_i'. However, such correlations, in which $\sigma = \rho'/\langle\rho\rangle$ is used instead of ρ', are not necessarily small. Therefore in the general case

$$w_j' \frac{\partial w_i'}{\partial x_j} = \frac{\partial w_i' w_j'}{\partial x_j} - w_i' \frac{\partial w_j'}{\partial x_j} = \\ = \frac{\partial w_i' w_j'}{\partial x_j} + w_i' \frac{\partial \sigma}{\partial t} + \langle w_j \rangle w_i' \frac{\partial \sigma}{\partial x_j} + w_i' \frac{\partial \sigma w_j'}{\partial x_j}, \tag{2.1}$$

and similarly for the corresponding averaged quantities. This relation is more exact than that used in [2]

$$\left\langle w_i' \frac{\partial w_j'}{\partial x_j}\right\rangle = 0, \quad \left\langle w_j' \frac{\partial w_i'}{\partial x_j}\right\rangle = \left\langle \frac{\partial w_i' w_j'}{\partial x}\right\rangle.$$

At point A in space the equations for the velocity fluctuations corresponding to the flow equations (1.2) take the form

$$\frac{\partial}{\partial t}(v_i)_A + [\langle v_j\rangle + (v_j)_A]\left(\frac{\partial}{\partial x_j}\right)_A (v_i)_A = \\ = -\frac{1}{d_1}\left(\frac{\partial}{\partial x_i}\right)_A p_A + \nu\left(\frac{\partial^2}{\partial x_j \partial x_j}\right)_A (v_i)_A - \frac{1}{d_1}(f_i)_A,$$

$$\frac{\partial}{\partial t}(w_i)_A + [\langle w_j\rangle + (w_j)_A]\left(\frac{\partial}{\partial x_j}\right)_A (w_i)_A = \frac{\varkappa}{d_1}\left(\frac{f_i}{\langle\rho\rangle}\right)_A.$$

Here and henceforth the prime that previously denoted fluctuation quantities has been dropped; as before averaged quantities are denoted by $\langle v_i\rangle$, $\langle w_i\rangle$ etc.

Multiplying the first of these equations by the value of the fluctuations of the j-th component of velocity at the point B and adding the result to the similar equation for the j-th component of velocity at the point B multiplied by $(v_i)_A$, we obtain, with account for the first of equations (1.3) and the fact that differentiation at the point A does not extend to $(v_j)_B$, and vice versa,

$$\frac{\partial}{\partial t}(v_i)_A (v_j)_B + \left(\frac{\partial}{\partial x_k}\right)_A (v_i)_A (v_k)_A (v_j)_B + \\ + \left(\frac{\partial}{\partial x_k}\right)_B (v_i)_A (v_k)_B (v_j)_B = -\frac{1}{d_1}\left[\left(\frac{\partial}{\partial x_i}\right)_A p_A (v_j)_B + \right. \\ \left. + \left(\frac{\partial}{\partial x_j}\right)_B p_B (v_i)_A\right] + \nu\left[\left(\frac{\partial^2}{\partial x_k \partial x_k}\right)_A (v_i)_A (v_j)_B + \right. \\ \left. + \left(\frac{\partial^2}{\partial x_k \partial x_k}\right)_B (v_i)_A (v_j)_B\right] - \frac{1}{d_1}[(f_i)_A (v_j)_B + (f_j)_B (v_i)_A]. \tag{2.2}$$

At $f_i = 0$, this equation coincides with the one usually employed [1].

Introducing the correlation notation

$$(V_{i,j})_{A,B} = \langle (v_i)_A \ (v_j)_B\rangle, \quad (K_{i,p}^{(v)})_{A,B} = \langle (v_i)_A p_B\rangle,$$
$$(K_{p,j}^{(v)})_{A,B} = \langle p_A (v_j)_B\rangle, (S_{ik,j}^{(v)})_{A,B} = \langle (v_i)_A (v_k)_A (v_j)_B\rangle, \tag{2.3}$$
$$(S_{i,kj}^{(v)})_{A,B} = \langle (v_i)_A (v_k)_B (v_j)_B\rangle,$$
$$(\Phi_{i,j}^{(v)})_{A,B} = \frac{1}{d_1}[\langle (f_i)_A (v_j)_B\rangle + \langle (f_j)_B (v_i)_A\rangle],$$

and the distance $\xi_i = (x_i)_B - (x_i)_A$ between A and B, as a result of averaging Eq. (2.2) we get

$$\frac{\partial}{\partial t} V_{i,j} + \frac{\partial}{\partial \xi_k}(S_{i,kj}^{(v)} - S_{ik,j}^{(v)}) = \\ = \frac{1}{d_1}\left(\frac{\partial}{\partial \xi_i} K_{p,j}^{(v)} - \frac{\partial}{\partial \xi_j} K_{i,p}^{(v)}\right) + 2\nu \frac{\partial^2}{\partial \xi_k \partial \xi_k} V_{i,j} - \Phi_{i,j}^{(v)}. \tag{2.4}$$

Quite analogously, taking into account (2.1), we get the following equation for the double correlation with respect to particle velocities:

$$\frac{\partial}{\partial t} W_{i,j} + \frac{\partial}{\partial \xi_k}(S_{i,kj}^{(w)} - S_{ik,j}^{(w)}) + L_{i,j}^{(w)} + \\ + \langle w_k\rangle M_{i,k,j}^{(w)} + N_{i,j}^{(w)} = \varkappa \Psi_{i,j}^{(w)}. \tag{2.5}$$

Here

$$(W_{i,j})_{A,B} = \langle (w_i)_A (w_j)_B\rangle,$$
$$(L_{i,j}^{(w)})_{A,B} = \left\langle (w_i)_A (w_j)_B \frac{\partial}{\partial t}(\sigma_A + \sigma_B)\right\rangle,$$
$$(M_{i,k,j}^{(w)})_{A,B} = \left\langle (w_i)_A (w_j)_B \frac{\partial}{\partial \xi_k}(\sigma_B - \sigma_A)\right\rangle, \tag{2.6}$$
$$(N_{i,j}^{(w)})_{A,B} = \left\langle (w_i)_A (w_j)_B \frac{\partial}{\partial \xi_k}[(\sigma w_k)_B - (\sigma w_k)_A]\right\rangle,$$
$$(\Psi_{i,j}^{(w)})_{A,B} = \frac{1}{d_1}\left[\left\langle\left(\frac{f_i}{\langle\rho\rangle}\right)_A (w_j)_B\right\rangle + \left\langle\left(\frac{f_j}{\langle\rho\rangle}\right)_B (w_i)_A\right\rangle\right],$$

and the correlations $S^{(w)}$ are expressed in terms of w_i just as $S^{(v)}$ are expressed in terms of v_i.

Using (1.3) and (2.1), it is also easy to obtain the equation for the dynamics of the change in the mixed correlation $\langle (v_i)_A (w_j)_B\rangle$. Adding the equation obtained by interchanging the pairs of subscripts i, j and A, B, we get

$$\frac{\partial}{\partial t} T_{i,j} + \frac{\partial}{\partial \xi_k}(S_{i,kj}^{(w,vv)} - S_{ik,j}^{(vv,w)}) + \frac{\partial}{\partial \xi_k}(S_{i,kj}^{(v,ww)} - S_{ik,j}^{(ww,v)}) + \\ + L_{i,j}^{(v,w)} + \langle w_k\rangle M_{i,k,j}^{(v,w)} + N_{i,j}^{(v,w)} = \frac{1}{d_1}\left(\frac{\partial}{\partial \xi_i} K_{p,j}^{(w)} - \right. \\ \left. - \frac{\partial}{\partial \xi_j} K_{i,p}^{(w)}\right) + \nu \frac{\partial^2}{\partial \xi_k \partial \xi_k} T_{i,j} - \Phi_{i,j}^{(w)} + \varkappa \Psi_{i,j}^{(v)}. \tag{2.7}$$

Here

$$(T_{i,j})_{A,B} = \langle (v_i)_A (w_j)_B\rangle + \langle (v_j)_B (w_i)_A\rangle,$$
$$(S_{i,kj}^{(w,vv)})_{A,B} = \langle (w_i)_A (v_k)_B (v_j)_B\rangle,$$
$$(S_{ik,j}^{(vv,w)})_{A,B} = \langle (v_i)_A (v_k)_A (w_j)_B\rangle, \tag{2.8}$$
$$(L_{i,j}^{(v,w)})_{A,B} = \left\langle (v_i)_A (w_j)_B \frac{\partial \sigma_B}{\partial t}\right\rangle + \left\langle (v_j)_B (w_i)_A \frac{\partial \sigma_A}{\partial t}\right\rangle,$$
$$(M_{i,k,j}^{(v,w)})_{A,B} = \left\langle (v_i)_A (w_j)_B \frac{\partial \sigma_B}{\partial \xi_k}\right\rangle - \left\langle (v_j)_B (w_i)_A \frac{\partial \sigma_A}{\partial \xi_k}\right\rangle,$$
$$(N_{i,j}^{(v,w)})_{A,B} = \left\langle (v_i)_A (w_j)_B \frac{\partial}{\partial \xi_k}(\sigma w_k)_B\right\rangle - \\ - \left\langle (v_j)_B (w_i)_A \frac{\partial (\sigma w_k) A}{\partial \xi_k}\right\rangle.$$

The correlations $K_{p,j}^{(w)}$, $K_{i,p}^{(w)}$, $\Phi_{i,j}^{(w)}$ (or $\Psi_{i,j}^{(v)}$) are expressed in terms of w_i (or v_i) in the same way as $K_{p,j}^{(v)}$, $K_{i,p}^{(v)}$, $\Phi_{i,j}^{(v)}$ (or $\Psi_{i,j}^{(w)}$) are expressed in terms of v_i (or w_i).

To make the above equations definite, we must find the representation of the correlations containing the force f_i in terms of the correlations with respect to fluid and particle velocities and the concentration ρ. For this purpose we must use the expression for f_i in terms of v, w and ρ (v and w are the total fluid and particle velocity vectors). Unfortunately, an analytical expression for the force developed by the fluid on a single particle can be given only if the fluid velocity varies little over distances large compared with the size of the particles, i.e., the spatial microscales of turbulence must be much greater than the particle size.

Here it is assumed that the force F developed by the fluid on a single particle is given by Stokes formula. Then for the force f_i we get

$$f_i = \frac{\rho}{\theta} F_i = c d_1 \rho (v_i - w_i), \qquad c = \frac{9\nu}{2a^2}, \tag{2.9}$$

where θ is the volume and a the radius of the particle.

In the general case the following expression has been proposed for F [8]:

$$\mathbf{F} = \frac{1}{2} d_1 \theta \left(\frac{\partial}{\partial t} + \mathbf{w}\cdot\nabla\right)(\mathbf{v} - \mathbf{w}) - \theta \nabla p - \tag{2.10}$$

$$- 6\pi\mu a \left[\mathbf{w} - \mathbf{v} + a \left(\frac{d_1}{\pi\mu}\right)^{1/2} \int\limits_{-\infty}^{t} \left(\frac{\partial}{\partial t} + \mathbf{w}\cdot\nabla\right)(\mathbf{w} - \mathbf{v}) \frac{d\tau}{\sqrt{t-\tau}}\right].$$

The first term in the right side is the excess intertia force for accelerated relative motion of the particle, the second is the excess pressure drop force, and the third the linear resistance force with allowance for the nonstationary effect.

The first and second terms can be neglected at $d_2 \gg d_1$. The integral term can be neglected if the particle size is sufficiently small.

Assumption (2.9) concerning the proportionality of the force f_i to the relative velocity $v_i - w_i$ is equivalent to the following assumptions which are fundamental to this study:

$$\rho \ll 1, \quad \varkappa = d_1 / d_2 \ll 1, \quad a \ll \lambda, \tag{2.11}$$

where λ is the internal scale of turbulence.

From the definition of the correlations $\Phi_{i,j}$ and $\Psi_{i,j}$, using (2.9), we get

$$(\Phi_{i,j}^{(v)})_{A,B} = c\langle\rho\rangle [2(V_{i,j})_{A,B} - (T_{i,j})_{A,B}] + cO_1(\langle\rho v w\rangle),$$
$$(\Phi_{i,j}^{(w)})_{A,B} = -c\langle\rho\rangle [2(W_{i,j})_{A,B} - (T_{i,j})_{A,B}] + cO_2(\langle\rho v w\rangle),$$
$$(\Psi_{i,j}^{(v)})_{A,B} = c[2(V_{i,j})_{A,B} - (T_{i,j})_{A,B}], \tag{2.12}$$
$$(\Psi_{i,j}^{(w)})_{A,B} = -c[2(W_{i,j})_{A,B} - (T_{i,j})_{A,B}].$$

Substituting (2.12) into Eqs. (2.4), (2.5), and (2.7), we obtain the system

$$\frac{\partial}{\partial t} V_{i,j} + \frac{\partial}{\partial \xi_k}(S_{i,kj}^{(v)} - S_{ik,j}^{(v)}) = \frac{1}{d_1}\left(\frac{\partial}{\partial \xi_i} K_{p,j}^{(v)} - \frac{\partial}{\partial \xi_j} K_{i,p}^{(v)}\right) +$$
$$+ 2\nu \frac{\partial^2}{\partial \xi_k \partial \xi_k} V_{i,j} - c\langle\rho\rangle(2V_{i,j} - T_{i,j}),$$

$$\frac{\partial}{\partial t} W_{i,j} + \frac{\partial}{\partial \xi_k}(S_{i,kj}^{(w)} - S_{ik,j}^{(w)}) + L_{i,j}^{(w)} + \langle w_k\rangle M_{i,k,j}^{(w)} + N_{i,j}^{(w)} =$$
$$= -c\varkappa(2W_{i,j} - T_{i,j}), \quad \frac{\partial}{\partial t} T_{i,j} + \tag{2.13}$$
$$+ \frac{\partial}{\partial \xi_k}(S_{i,kj}^{(w,vv)} + S_{i,kj}^{(v,ww)} - S_{ik,j}^{(vv,w)} - S_{ik,j}^{(ww,v)}) + L_{i,j}^{(v,w)} +$$
$$+ \langle w_k\rangle M_{i,k,j}^{(v,w)} + N_{i,j}^{(v,w)} = \frac{1}{d_1}\left(\frac{1}{\partial \xi_i} K_{p,j}^{(w)} - \frac{\partial}{\partial \xi_j} K_{i,p}^{(w)}\right) +$$
$$+ \nu \frac{\partial^2}{\partial \xi_k \partial \xi_k} T_{i,j} + 2c(\langle\rho\rangle W_{i,j} + \varkappa V_{i,j}) - c(\langle\rho\rangle + \varkappa) T_{i,j}.$$

Thus, in the case of a nonhomogeneous fluid the single equation for the velocity correlation $(V_{i,j})_{A,B}$ for a homogeneous fluid is replaced by three equations for the three double correlations $(V_{i,j})_{A,B}$, $(W_{i,j})_{A,B}$ and $(T_{i,j})_{A,B}$.

3. Decay of isotropic turbulence. As is known [1], for isotropic turbulence in an incompressible fluid correlations of the type $\langle g v_i\rangle$ (where g is any scalar) are identically equal to zero. In particular, $K_{i,p}^{(v)} \equiv K_{p,j}^{(v)} \equiv 0$. For a compressible fluid, from the condition of invariance with respect to spatial rotations it follows that

$$K_{i,p}^{(w)} = K_{p,i}^{(w)} = K(r)\,\xi_i$$

where r is the scalar distance between points A and B. Therefore, as it is easy to see,

$$\frac{\partial}{\partial \xi_i} K_{p,j}^{(w)} - \frac{\partial}{\partial \xi_j} K_{i,p}^{(w)} = 0.$$

It is also easy to show that

$$M_{i,k,j}^{(w)} \equiv M_{i,k,j}^{(v,w)} \equiv 0.$$

(this is immediately clear from Eqs. (2.13); the terms with $M_{i,k,j}$ containing "nonisotropic" multipliers $\langle w_k\rangle$, must vanish in describing isotropic turbulence). Moreover, for the double and triple correlations in (2.13) from considerations of general invariance there follow the known relations that enable these correlations to be represented in terms of a small number of scalar functions of r and time [1]. All this makes it possible considerably to simplify (2.13).

Below we shall discuss in detail only the case where the influence of the viscous and interaction forces becomes predominant as compared with the inertia forces ("final decay period" to use the terminology of Batchelor and Townsend [9]). Then, in accordance with the general principle, we can neglect the triple correlations in (2.13). Contracting (2.13), in this case we get the following system for the traces of the correlation tensors:

$$\frac{\partial}{\partial t} V_{i,i} = 2\nu \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} V_{i,i}\right) - c\langle\rho\rangle(2V_{i,i} - T_{i,i}),$$
$$\frac{\partial}{\partial t} W_{i,i} = -c\varkappa(2W_{i,i} - T_{i,i}),$$
$$\frac{\partial}{\partial t} T_{i,i} = \nu \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} T_{i,i}\right) - \tag{3.1}$$
$$- c(\langle\rho\rangle + \varkappa) T_{i,i} + 2c(\langle\rho\rangle W_{i,i} + \varkappa V_{i,i}).$$

We shall first consider the case of high-inertia particles: $\varkappa \to 0$. From the second of Eqs. (3.1) it follows that $W_0 = W_{i,i} = W(r)$. In this case only damping solutions of system (3.1) make physical sense; therefore $W_0 = 0$. Thus, in the last of Eqs. (3.1) we are left with only one unknown $T_{i,i}$. However, it is not necessary to solve this equation. In fact, at r = 0, W_0 must be equal to $3w_0^2$, where

$$w_0^2 = \langle w_1'^2\rangle \equiv \langle w_2'^2\rangle \equiv \langle w_3'^2\rangle.$$

Since $W_0 = 0$ everywhere, the quantity $w_0 = 0$ and the fluctuations $w_i' \equiv 0$. Hence and from the definition $T_0 = T_{i,i}$ it follows that T_0 is identically equal to zero. (The physical meaning of this is obvious: the parameter $\varkappa = 0$ corresponds to infinite particle density, so that the fluctuation motions of the fluid have no effect on the particle velocity, which at any moment of time is equal to its mean. Within the limits of a single fluctuation the situation resembles the motion of a fluid in an evacuated porous body.) Of course, the same conclusion may be reached directly from a consideration of the equations of motion as $\varkappa \to 0$.

Consequently, for $V_0 = V_{i,i}$ we get the equation

$$\frac{\partial V_0}{\partial t} = 2\nu \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} V_0\right) - 2c\langle\rho\rangle V_0 . \qquad (3.2)$$

As is known [7], in the case of uniform turbulence the boundary conditions imposed on the solutions of the corresponding equations (in particular, Eq. (3.2)) are determined by the existence of statistical uniformity of motion in space and do not require further examination. The initial conditions consist in that at a certain instant the velocity is a random function of the point. The actual form of this function is unknown and only the average quantities characterizing the turbulence field at the initial moment are given [7].

Here we take

$$\lim_{r\to 0} V_{i,i}(r, t) = 3v_0^2 \equiv 3\langle v_1'^2\rangle \equiv 3\langle v_2'^2\rangle \equiv 3\langle v_3'^2\rangle, \qquad (3.3)$$

it being assumed that, as usual, the mean square of the one-dimensional fluctuation velocity v_0^2 must also be determined from the solution (3.2).

Moreover, from the continuity equation for an incompressible fluid, with certain additional assumptions, it follows [1] that

$$\int_0^\infty r^2 V_{i,i}(r, t)\, dr = 0 . \qquad (3.4)$$

Conditions (3.3) and (3.4) are the only constraints that the general theory permits us to impose on solutions of (3.2).

It is easy to see that

$$V_0(r, t) = \exp(-2c\langle\rho\rangle t)\, Q_{i,i}(r, t) , \qquad (3.5)$$

where $Q_{i,i}(r,t)$ is the double velocity correlation in the homogeneous fluid, satisfying (3.2) without the last term on the right side. Taking as $Q_{i,i}(r,t)$ the known solution of M. D. Millionshchikov [10], which is in good agreement with experiment [9], we get

$$V_0(r, t) = -4C_2 t^{-5/2}\left(3 - \frac{r^2}{4\nu t}\right)\exp\left(-\frac{r^2}{8\nu t} - 2c\langle\rho\rangle t\right). \qquad (3.6)$$

Thus, the intensity of the fluctuations decays in accordance with the law

$$v_0^2 \sim t^{-5/2}\exp(-2c\langle\rho\rangle t) . \qquad (3.7)$$

Hence it is clear that the decay of turbulence in a nonhomogeneous fluid differs very strongly from that in the homogeneous fluid. The law of $-5/2$ characteristic of the homogeneous fluid is replaced by an exponential decay conditioned by the dissipation of fluctuation energy on the suspended particles. For $V_{i,i}$ we can write [1]

$$V_{i,i} = v_0^2 [f(r, t) + 2g(r, t)] = \frac{v_0^2}{r^2}\frac{\partial}{\partial r}[r^3 f(r, t)] .$$

Using $V_{i,i}(r,t)$ from (3.6), for the longitudinal $f(r,t)$ and lateral $g(r,t)$ correlation coefficients we get

$$f(r, t) = \exp\left(-\frac{r^2}{8\nu t}\right), \qquad g(r, t) = \left(1 - \frac{r^2}{8\nu t}\right)\exp\left(-\frac{r^2}{8\nu t}\right). \qquad (3.8)$$

These expressions completely coincide with the expressions for f and g in the case of turbulence in a homogeneous fluid. Moreover, the space scales of turbulence, determined from a study of the correlations, are completely defined by these coefficients. Therefore, in the nonhomogeneous fluid considered, as $\varkappa \to 0$ the characteristic dimensions of the eddies are identical to those in the homogeneous fluid.

The three-dimensional spectrum function

$$E(k, t) = \frac{1}{\pi}\int_0^\infty kr \sin kr\, V_{i,i}(r, t)\, dr = \varepsilon^\circ(k, t)\exp(-2c\langle\rho\rangle t), \qquad (3.9)$$

where $\varepsilon^\circ(k,t)$ is the corresponding spectrum function for turbulent motion of a homogeneous fluid.

Thus, the turbulent motion of a suspension of high-inertia particles in an incompressible fluid is, in the final period of decay of turbulence, similar in structure to the turbulent motion of the homogeneous fluid. The only difference is in the more rapid decay of the fluctuations in the first case as compared with the second (damping of turbulence by the particles), while, as follows from (3.8) and (3.9), the effect of the particles becomes predominant at large t.

This conclusion is not unexpected. It is usual to assume [1] that since the additional dissipation of fluctuation energy is conditioned by "slip" of the particles due to the turbulent motion of the fluid and since this slip increases with increase in the wave number of the turbulence, the presence of particles must exert an effect on the energy spectrum of the turbulence mainly in the region of high wave numbers. However, in the particular case examined the particles are actually stationary, i.e., their slip with respect to the fluid does not depend on the wave number, as a result of which there is no distortion of the structure of the turbulence energy spectrum.

As $\varkappa \to 0$ it is also easy to obtain a more general equation for V_0—with allowance for the triple velocity correlations. Setting, as before, $w_0^2 \equiv 0$ and $W_0 \equiv T_0 \equiv 0$, we write the corresponding equation of the Karman-Howart type [11]. For this purpose we use the known expression of the theory of isotropic turbulence in an incompressible fluid [1]

$$S_{i,j} = \frac{\partial}{\partial \xi_k}(S_{ik,j}^{(v)} - S_{j,kj}^{(v)}) = v_0^3\left[\left(-\frac{1}{2r}\frac{\partial^2 k}{\partial r^2} - \frac{2}{r^2}\frac{\partial k}{\partial r} + \frac{2}{r^3}k\right)\xi_i\xi_j + \left(\frac{r}{2}\frac{\partial^2 k}{\partial r^2} + 3\frac{\partial k}{\partial r} + \frac{2}{r}k\right)\delta_{ij}\right],$$

$$k = k(r, t).$$

Substituting this and the expression for $V_{i,j}$ in terms of $f(r,t)$ in the first of Eqs. (2.13), after transformations and integration with respect to r we get

$$\frac{\partial}{\partial t}(v_0^2 f) - v_0^3 \frac{2}{r^4}\frac{\partial}{\partial r}(r^4 k) = 2\nu v_0^2 \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial f}{\partial r}\right) - 2c\langle\rho\rangle v_0^2 \frac{1}{r^3}\frac{\partial}{\partial r}(r^3 f) . \qquad (3.10)$$

For $\langle\rho\rangle = 0$ this equation reduces to the usual Karman-Howart equation.

The stabilizing action of the suspended particles and the decrease in fluctuation energy in unit volume of mixture as compared with the fluctuation energy of the homogeneous fluid were established in [2] on the basis of the fluctuation energy balance. As might have been expected, the reverse effect, i.e., excitation of turbulence by the suspended particles, was not observed. This is connected with the assumption that the gravitational forces are small and that $\langle v_i \rangle \equiv \langle w_i \rangle$.

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